## Universitat Politècnica de Catalunya - BarcelonaTech

# IMTECH $\quad 6 a$ <br> Newsletter 

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Eva Miranda Galcerán ${ }^{\text {® }}$ is a Full Professor at UPC ${ }^{\text {e }}$ ，a mem－ ber of IMTech ${ }^{『}$ ，and a member of $\mathbf{C R M}^{\complement}$ ．She is the director of the Lab of Geometry and Dynamical Systems ${ }^{\text {® }}$ and group leader of the Geometry group ${ }^{5}$ at UPC．Distinguished with two ICREA Academia ${ }^{\text {® }}$ Prizes in 2016 and 2021，she was awarded a Chaire d＇Excellence de la Fondation Sciences Mathématiques de Paris ${ }^{\circledR}$ in 2017 and a Bessel Prize ${ }^{\circlearrowright}$ in 2022．She has also been the recipient of the François Deruyts ${ }^{『}$ Prize，a quadren－ nial prize conferred by the Royal Academy of Belgium，in 2022. In 2023 she was Hardy Lecturer ${ }^{\text {® }}$ by invitation of the London Mathematical Society ${ }^{\text {E }}$ ．

Miranda＇s research is at the crossroads of Differential Ge－ ometry，Mathematical Physics and Dynamical Systems．In the last years，she added to her research agenda mathematical as－ pects of theoretical computer science in connection to Fluid Dynamics．

A decade ago she pioneered the investigation of $b$－Poisson manifolds．These structures appear naturally in physical sys－ tems on manifolds with boundary and on problems on Celestial Mechanics such as the 3－body problem．

In 2021 she constructed（jointly with Robert Cardona ${ }^{\text {®，}}$ ， Daniel Peralta Salas ${ }^{〔}$ ，and Francisco Presas ${ }^{\text {® }}$ ）a Turing com－ plete ${ }_{3} \mathrm{D}$ Euler flow．This result not only proves the existence of undecidable paths in hydrodynamics，but also closes an open question in the field of computer science（the existence of＂fluid computers＂）．

Miranda＇s research strives to decipher the several levels of complexity in Geometry and Dynamics．She endeavors to ex－ tend Floer homology and the singular Weinstein conjecture to the singular set－up motivated by the search of periodic orbits in Celestial Mechanics．

NL．This NL has tried to promptly echo your research achieve－ ments and a number of concomitant recognitions since its incep－ tion in January 2021．They are summarized in the various Edi－ torial pieces，with pointers to the details in the inner pages．For instance，the last issue（ $\mathrm{NLO}_{5}{ }^{\text {® }}$ ）included a report on the first two parts of your Hardy Tour lectures，which by now they have been completed with the last two（19 and 21 September），but you surely have been much active in other endeavors as well．To begin with， however，we would like to go back a few years．Did it all start somehow with your ICREA Academia in 2016？

The ICREA Academia has been a unique opportunity for me to
focus on research at a dream level．ICREA has enabled me to take risks in my research which in turn yielded results that have had an important impact inside and outside mathematics．So I must say ICREA opened the door to what came after and I am forever grateful to have had such a great chance back in 2016．The honor coincided in time with a Chaire d＇Excellence ${ }^{『}$ of the Fondation des Sciences Math－ ématiques ${ }^{\complement}$ de Paris．I was also honored with an ICREA Academia 2021 which has enabled me to pursue this intensification of research． In practice，this means that my teaching has been reduced to one master course and the rest of the time should be devoted to research and administration（but mostly research）．I also have gathered a big group around me．Since 2016， 6 PhD students have defended their doctoral thesis under my supervision and currently I am advising 5 more．ICREA Academia gives me the freedom to sail my research in the direction that I want with long－term very ambitious projects．
You also got the François Deruyts prize in Geometry conferred by the Royal Academy of Belgium，which is a great achievement．Can you tell us more about it and what represents in your career？

In 2022 I got the François Deruyts prize in Geometry which is conferred by the Royal Academy of Belgium every four years．This is a unique distinction，as I am the only Spanish person in the list of awardees，and in fact the only non－Belgium awardee．The François Deruyts Prize，also known as the Prix Francois Deruyts，is presented once every four years to acknowledge advancements in the fields of synthetic or analytic superior geometry．This esteemed award was founded in 1902 by the Académie Royale de Belgique，specifically by its Classe des Sciences，and includes a monetary award．The list in－ cludes names such as Jacques Tits ${ }^{\complement}$ and Pierre Deligne ${ }^{\complement}$ or Michael Cahen and Simone Gutt ${ }^{\text { }}$ ．I feel very honored to be on such a list． The award ceremony took place in the Palais de l＇Académie ${ }^{\circledR}$ in Brus－ sels in December 2022．This was a high moment on my career with very emotive words by the Sécretaire Perpetuelle de l＇Académie Royale， Didier Viviers ${ }^{\text {® }}$ ．

## The Humboldt Foundation conferred you a Bessel prize．Can you tell us more about this prize？

Yes，I also got a Bessel Prize conferred by the Humboldt Foun－ dation ${ }^{\complement}$ ．This second recognition came with homework．I have to stay at least 6 months in Germany to foster collaboration with sev－ eral institutions．In my case，it was the Universität zu Köln ${ }^{\mathbb{}}$ who nominated me for this award and this is the main institution of my stay．Other stops are Universität Augsburg ${ }^{\complement}$ ，Universität Erlangen－ Nürnberg ${ }^{\complement}$ ，Ruhr－Universität Bochum ${ }^{\complement}$ ，Universität Heidelberg ${ }^{\complement}$ ，and Universität Göttingen ${ }^{\text {T }}$ ．As I decided to use the Humboldt Prize to create a new network of collaborations，the research visits will be ex－ tended until 2025．As an offspring of the visits this year，we are apply－ ing for a Collaborative Network that involves the Universität Augsburg， the UPC $^{\text {® }}$ ，and ICMAT ${ }^{\complement}$ in Spain．My collaboration with Cologne has three different directions：that of contact geometry with Professor Hansjörg Geiges ${ }^{\text {T }}$ ，that of Quantization with Professor George Mari－ nescu ${ }^{\text {® }}$ ，and that of Toric manifolds with professor Silvia Sabatini ${ }^{『}$ ． The collaboration with the three subgroups has been fostered through several seminars and workshops．

Let us now focus on the Hardy lectures 2023．Could you assess what they represent for your career？Could you also comment on why your tour includes more lectures than in any preceding edi－ tion？How have you managed such a dense program？

To be named the 2023 Hardy lecturer is a momentous recognition of my career．This extraordinary award stands as an unparalleled
milestone in my professional journey．Former Hardy lecturers include acclaimed mathematicians such as Dusa McDuff ${ }^{〔}$ ，Terence Tao ${ }^{\text {® }}$ ，Yu Manin ${ }^{\complement}$ ，Etienne Ghys ${ }^{\complement}$ ，Jacob Lurie ${ }^{\complement}$ ，Nalini Joshi ${ }^{\text {® }}$ and Peter Sar－ naK ${ }^{[\nearrow}$ ，just to mention a few．Again this nomination came with the commitment to deliver several lectures around the UK，In principle the prevision were six lectures，plus the one at the general meeting of the LMS，but at the end they were nine．The reason for this increase is that once the tour was announced，two more institutions asked me for additional tals：The Royal Institution of London ${ }^{\complement}$ and the University of Warwick ${ }^{\boxed{ } 1}$ ．

We would also like to get your views on the various institutions in which the lectures have been delivered．In particular，we would like to know where and when the picture on the＂Penrose way＂was taken？

The tour has given me a global vision of several institutions throughout the UK．Most importantly，I have connected with many interesting individuals of diverse origin：mathematicians，physicists and computer scientists and now I am ready to start new research adventures with some of them．One of them is Roger Penrose ${ }^{\complement}$ with whom we are currently revisiting some of the basics of Twistor theory and some mysterious symmetry that breaks into the theory and was not accounted for before．With Professor Raymond Pierrehumbert ${ }^{\text {C® }}$ we would like to understand new connections between $b$－symplectic geometry and the detection of exoplanets．It has been the perfect adventure in all possible ways．The picture＂The Penrose way＂was taken at the University of Loughborough ${ }^{\complement}$ campus while entertaining new interesting connections and adventures in mind：I want to go ＂the Penrose way＂．
In your tour you have met many people that have been involved in hosting your lectures and in organizing additional activities． Would you mind sharing your views on these aspects of the tour？

I have been absolutely delighted with my hosts．In each institution there was，besides several contacts，an official host．The hosts took care of organizing the activities and official reception at each univer－ sity．At each stop there was also an official dinner．This gave me the opportunity to socialize with many different people．
Since for a researcher there is no resting on one＇s laurels，we would much appreciate if you could describe in some detail your goals for the next few years and some activities you are envisioning to achieve them．

The Fluid computer stroke my mind as a revelation but was insuf－ ficient for the purpose of finding blow－up solutions of PDE＇s．So in order to achieve that we need to make the theory more complete．In a way our Fluid computer was not enough：One of the aspects that has completely taken my attention the last months is the design of a new model of theoretical computer．I am currently working on a hybrid machine between Fluid and Quantum computer．I do this fol－ lowing Topological Quantum field theory in collaboration with Angel Gonzalez－Prieto ${ }^{\complement}$ and Daniel Peralta Salas ${ }^{\complement}$ ．Another aspect is the applications of my theory to detection of escape orbits connected to several long－standing conjectures in Symplectic Topology．One of the works that I am recently pursuing goes in the direction of disproving one of these conjectures．More soon！The last couple of months I had a couple of interesting surprises．On the one hand，I have been nominated by the Universität zu Köln as the Mercator professor ${ }^{〔}$ ．This professorship has the role of ambassador of the University of Cologne in the world．This recognition is a source of immense joy for me，as it reflects my strong and interconnected international relationships．I have also been invited to teach a Nachdiplom course ${ }^{『}$ at ETHZ in Zurich in the Fall of 2025．More details soon！

As an IMTech member，how do you see its future？In your view， what synergies should be promoted between the various stakehold－ ers，internal and external，in order to optimally fulfill its vision？

The IMTech has been consolidated in a very interesting moment for mathematics in Barcelona．It arrives in the right moment！Unlike other institutions in Catalonia，IMTech has a differential trait：that of gathering more interdisciplinary projects common with researchers closer to Engineering and Computer Science．Diversity is our distinct flag that makes us so special．I am thrilled to be part of it．There is a lot of work to do and it would be a good idea to organize more activities to encourage cross－fertilization among different disciplines reflected in its composition．The IMTech needs to position itself as an strategic＂lighthouse＂in this stunning city by the sea，Barcelona， that can attract international talent from all the cardinal points．Our unique character makes us so special that we are almost＂irresistible＂ as a trademark．In my opinion we need to intensify our internation－ alization aspects and make our dream bigger．However，substantial dreams require tangible backing from financial institutions．IMTech requires further support to make our dream come true．


A mathematical proof should resemble a simple and clear－cut constellation， not a scattered cluster in the Milky Way．
G．H．Hardy（1992）．A Mathematician＇s Apol－ OGY


Without dreams there is no art，no mathematics，no life．
Sir Michael
Atiyah（2010），
Notices of THE AMS



Guadalupe Gómez Melis ${ }^{〔}$ is a Professor at the Universitat Politècnica de Catalunya－BarcelonaTECH（UPC ${ }^{『}$ ）．She leads the Research group on Biostatistics and Bioinformatics GRBIO UPC－ UB ${ }^{\text {® }}$ ．She has been visiting scientist at Harvard University ${ }^{\text {® }}$ （Boston，USA），Oxford University Clinical Research Unit（Ho Chi Minh City，Vietnam）and the MD Anderson Cancer Cen－ ter（Houston，USA）．She received the Bachelor and PhD degrees in Mathematics from the Universitat de Barcelona（UB ${ }^{\text {CB }}$ ）and MSc and PhD degrees in Statistics from Columbia University （NY，USA）．She is president of the Consell Català d＇Estadística ${ }^{\text {® }}$ ， representative of the Catalan universities at the Consell de Salut de Catalunya ${ }^{\text {® }}$ ，member of the UPC Ethics Committee ${ }^{\text {® }}$ ， elected member of the Council of the International Biomet－ ric Society ${ }^{\text {E }}$ ．Former vice－dean in the School of Mathematics and Statistics（FME ${ }^{\text {C }}$ ），founder and coordinator of the Master in Statistics and Operations Research UPC－UB ${ }^{\text {® }}$ ，coordinator of the Interuniversity Doctorate in Bioinformatics ${ }^{\complement}$ and coordinator of the Doctorate in Statistics and Operations Research ${ }^{\complement}$ ，elected European Representative of the Caucus for Women in Statis－ tics ${ }^{『}$ ，and a recipient of The Marvin Zelen Memorial Lecture of EMR－IBS ${ }^{巳 〕}$ ，an award recognizing and honoring her influence in the field of Biostatistics．Her main research interest is in devel－ oping methods for Survival Analysis and Clinical Trials ${ }^{\text {® }}$ ，with an unequivocal interdisciplinary flavor，focusing especially on cancer，HIV－AIDS and lately on COVID－19．She is the PI of sev－ eral funded projects of the Ministerio de Ciencia e Innovación and the Generalitat de Catalunya．

NL．You did your Bachelor in Mathematics at the UB．How did you decide to do a PhD and，in particular，to go to the USA to do $i t$ ？

After doing the Tesina de Licenciatura（Bachelor Thesis），directed by Prof．David Nualart ${ }^{\text {एँ }}$ ，on Stochastic Processes，I realized that I loved research but，if possible，connected to real world problems．This is why I started the PhD at the UB in Statistics．I did all the courses of the two first years but when looking for thesis subjects I was not engaged by any of the research topics that I was proposed．Then，ev－ erything started as a game；I was only 23 years old and I was fearless， adventurous and a hard worker，and we（with my partner since then， Àlvar Vinacua）started to fill endless documents，send them by regular mail，waiting for the answers，and in the meantime took the required
exams for the US Graduate Programs，such as the Graduate Record Ex－ amination．The acceptance at the Statistics Department at Columbia University ${ }^{[3}$ in New York and the scholarship from the Generalitat de Catalunya did the rest and on August 1982 we landed in JFK．

## When did you know that you wanted to work in Statistics？

I liked very much the theoretical aspects of Mathematics but I was attracted to real life problems and Statistics was the perfect combina－ tion．

I＇m sure that during your years at the USA，in addition to the theory and practice of Statistics，you learned a lot about doing and managing research，or university organization．Could you mention some of these non－strictly－academic skills you are most grateful for？

I am very grateful to those years in New York，later in Ohio State University ${ }^{\text {E }}$ as Assistant professor，because I learned a lot about what a research department meant，how important was to keep a good atmosphere among professors and students，the generosity and hum－ bleness of professors．I was lucky enough to learn the importance of collaboration and the real meaning of interdisciplinarity，how close you should be to the real problem and how much you can learn from other colleagues．

Your main research topic is survival analysis．Can you tell us how you came to this topic？What do you consider your most relevant contributions in survival？

While taking PhD courses at Columbia，I fell in love with the sur－ vival analysis course given by Professor John Van Ryzin ${ }^{〔}$ ，who became my thesis advisor．This topic gathered my two ambitions：challenging statistical problems and relevant real－life scientific problems．Along the years there are two research lines where I have made relevant contributions．I have worked a lot on interval censoring，and in this field our group is now a referent．The beauty of this research line is that we have fundamental papers on the topic of noninformativeness， along many papers developing new methods for different problems and finally highly cited contributions in the Shelf Life area（within Food Technology）where our methods have become the standard for their analysis．A second topic refers to Composite endpoints in Clin－ ical Trials．With this topic，I approached clinical trials in a different way，and took me through different avenues，new collaborators and last，but not least，web app tools to make our methods usable by oth－ ers．Both topics have been followed by many of my PhD students and thanks to them they are alive．I am very grateful to professor Steve Lagakos ${ }^{『}$ from Harvard University with whom I started to collaborate in 1989 and together we started the two previous topics．

## On the side of teaching and teaching organization，how would you describe your main contributions and achievements？

In this aspect the Interuniversity Master in Statistics and Operations Research（MESIO UPC－UB）has been my main achievement．Back in 2006 when new regulations geared the Spanish universities to de－ velop official Master degrees，the FME trusted me to lead a team to develop the new curriculum．Even with all the limitations we had， we put together an ambitious program where interdisciplinarity had to be present，students from different backgrounds were going to sit together and applications were going to have as much respect as theo－ retical subjects．During 10 years I coordinated this master which after 2 years became a joint program with the UB Universitat de Barcelona， which turned out to be a relevant and smart decision．

Over the years, you have been able to create a very active, dynamic and powerful research group. Can you summarize the evolution of this team, its greatest achievements and its projection in the medium term?

Thanks to my years in US when I returned to Catalunya I was not attached to any particular group or university and I interacted with colleagues from all the institutions. This helped a lot and was first materialized into the GRASS (Grup de Recerca en Anàlisi eStadistic de la Supervivència) that started in 1995 and became the research seed for many of today's senior researchers. Later, with the Generalitat de Catalunya call to give support to the scientific activities of the Catalan research groups, the GRASS was split, mainly by institutions, and we put together the GRBIO UPC-UB (Grup de Recerca en Biostadística i Bioinformàtica). Since 2014 we have been growing and thanks to the Generalitat funding ( 2014 SGR 464, 2017 SGR 622 and 2021 SGR 01421) we have grown not only in number ( 23 investigators and 6 PhD students) but more importantly in quality and achievements. Our biweekly seminars, our software contributions, our outreach involvement together with a fair play and mutual support has facilitated our research as well as its impact and the large number of publications. The members of GRBIO are as well members of the Catalunya-BIO node of the BIOSTATNET ${ }^{\complement}$ (Spanish network of Biostatistics). Together with seven professors from different Spanish universities we founded Biostatnet in 2010 and since then it has grown and become, together with the SEB (Sociedad Española de Bioestadística), the place of confluence for researchers in biostatistics in Spain and the place of training and the starting point of the scientific career of most of our young people. I have been very lucky to have been surrounded by a very friendly, intelligent and active biostatistics community, that has evolved into a very cohesive group; hence, I foresee a long and productive life ahead for the GRBIO and BIOSTATNET. Since July 2023 GRBIO joined the Research Centre for Biomedical Engineering (CREB $\mathrm{UPC}^{\text {© }}$ ) and is one of the research group partners of the network Xartec Salut led by CREB UPC.

## You have participated in several mentoring initiatives. Can you tell me about your experience?

I love mentoring and I think it should be a relevant part of our duties as academics. The advantages of mentoring both for the mentees and for the mentors themselves are countless. Mentoring contributes to the personal, academic, and professional growth of the mentees. It helps them build a solid foundation for a successful and fulfilling career in academia and beyond. As mentors we can share our expertise, experiences, and knowledge helping younger scholars to gain insights that might not be readily available through formal education. Witnessing the growth and success of mentees can be deeply rewarding.

Mentoring young scholars is an investment in the future of academia and society at large. Among the several mentoring initiatives in which I have participated, the one we launched between the Sociedad Española de Bioestadística and the network BIOSTATNET is being very successful and has been the seed for the International Biometric Mentoring program within the International Biometric Society in which I have also collaborated to launch it.

Recently you have been involved in a research project call DIVINE ${ }^{\text {C, at }}$ which you have analyzed COVID-19 in Catalonia from a biostatistical perspective. Could you summarize your main findings?

The project Dynamic evaluation of COVID-19 clinical states and their prognostic factors to improve intra-hospital patient management (DIVINE) was funded under the call Pandèmies 2020 of the Generalitat de Catalunya. This project had four main goals and all of them were achieved after 18 months. Data of 5,813 hospitalized adult patients with confirmed COVID-19 in 5 hospitals (Barcelona South Metropoli$\tan$ area) and corresponding to four waves of the pandemic between March 2020 and August 2021 was collected including dates to enter into different stages (severe and non-severe pneumonia; invasive and noninvasive mechanical ventilation; recovery; discharge and death) together with demographic data, comorbidities, vital signs, laboratory results, and previous medications. Using multi-state models (MSM), the generalized odds-rate class of regression models and clustering techniques, i) we identified the most clinically relevant prognostic factors and in particular that low levels of the ratio of oxygen saturation to the fraction of inspired oxygen, and high concentrations of the C-reactive protein, were risk factors for health deterioration among hospitalized COVID-19 patient; also that patients with at least one vaccination dose were slightly better-off, but mainly to prevent severity at earlier stages; ii) we developed the App MSMpred allowing the interpretation of multi-state models and prediction of the disease course of future patients; iii) we estimated the incubation time period of the COVID-19 with data from the fifth wave pandemic learning that the estimated median Sars-Cov-2 incubation period was 2.8 days ( $95 \%$ CI: $[2.5,3.1]$ days) and no statistically significant differences were found when comparing vaccinated versus unvaccinated patients; and iv) patients' profiles can be clustered between waves 1-3 and wave 5 and vaccination was crucial to distinguish among three clusters found in wave 5 . The fruitful collaboration between statisticians and clinicians has been key in developing a model for the disease course of hospitalized COVID-19 patients at a higher risk of developing severe outcomes. Besides the acquired knowledge about the disease, the existing and the new developed methodology applied in this project sets the foundations for further analysis and management of hypothetical future pandemics.

## Research focus

Multiview varieties: a bridge between Algebraic Geometry and Computer vision,
by Angélica Torres ${ }^{『}$ (Maria de Maeztu fellow at $\mathrm{CRM}^{『}$ )
Received September 18, 2023
Computer Vision is the area of Artificial Inteligence that focuses on computer perception and processing of images. One of the main problems that arise in this area is the so called Structure-from-motion (SfM) problem or 3D-image reconstruction problem, where the main goal is to create a $3_{3} \mathrm{D}$ model of an object appearing in multiple two-dimensional images.

The input of the problem is a data set of images and the output is a $3^{D}$ model of the scene consisting of an estimation of the relative position of the cameras and a relative position of the objects in the pictures with respect to the cameras.

Solving this problem as accurately and quickly as possible is fundamental for autonomous driving, videogames, and animation, just to name a few examples.

The SfM pipeline is depicted in Figure 1 and it has four key steps:

## 1. Data collection.

2. Matching. In this step we match points or lines in one image that are identifiable as the same point or line in another. These pairings are called correspondences.
3. Camera pose. In this step, some point and line correspondences are used to estimate the relative position of the cameras.
4. Triangulation. The camera positions are used to estimate the position of the world and line points whose images are the point and line correspondences obtained in the matching step.

The final ${ }_{3} \mathrm{D}$ model is given by the camera positions and the world points and lines obtained from steps 3 and 4 .

Although the input of the SfM pipeline is a set of images, at the end of Step 2 the images are forgotten and we are left with purely geometric information: correspondences of points or lines that are believed to come from the same world feature. To analyse this geometric information, the use of Algebraic Geometry is a natural choice.

## Algebraic models for computer vision

In this note we will focus on algebraic varieties appearing in the Triangulation step. To understand them we start by modeling the process of taking a picture.

A camera is a function $C: \mathbb{P}^{3} \rightarrow \mathbb{P}^{2}$. Depending on the camera that we want to model, the definition of this function
is going to change. A pinhole camera $C$ is a linear projection from $\mathbb{P}^{3}$ to $\mathbb{P}^{2}$ defined by a $3 \times 4$ matrix of full rank. For this model we assume that it is possible to take pictures of every point in space except for $c=\operatorname{ker}(C)$, which is called the camera center and, intuitively, represents the position of the camera in the world.
Given $m$ pinhole cameras $C_{1}, \ldots, C_{m}$, the joint camera map is defined as

$$
\begin{align*}
\varphi_{\mathcal{C}}: & \mathbb{P}^{3} \longrightarrow\left(\mathbb{P}^{2}\right)^{m} \\
& X \longmapsto\left(C_{1} X, \ldots, C_{m} X\right) . \tag{1}
\end{align*}
$$

This map models the process of taking the picture of a world point with $m$ cameras, that is, for each $X \in \mathbb{P}^{3}$ the tuple $\varphi_{\mathcal{C}}(X) \in\left(\mathbb{P}^{2}\right)^{m}$ is a point correspondence.

Using the camera map, we can define similar functions to model line correspondences. Given a camera $C$, and two points $u, v \in \mathbb{P}^{3}$, denote by $L_{u, v}$ the line spanned by $u$ and $v$. The map

$$
\begin{gather*}
v_{C}: \operatorname{Gr}\left(1, \mathbb{P}^{3}\right) \longrightarrow \operatorname{Gr}\left(1, \mathbb{P}^{2}\right) \\
L_{u, v} \longmapsto L_{C u, C v}, \tag{2}
\end{gather*}
$$

where $\operatorname{Gr}\left(1, \mathbb{P}^{n}\right)$ denotes the grassmannian of lines in $\mathbb{P}^{n}$, takes a line spanned by $u$ and $v$ and sends it to the line in $\operatorname{Gr}\left(1, \mathbb{P}^{2}\right)$ spanned by $C u$ and $C v$. This map models the process of taking the picture of a line with a camera $C$. It is straightforward to prove that the definition of the map does not depend on the choice of $u$ and $v$, so from now on we can omit these subindices from the notation. Since $C$ is not defined in the camera center, the map $v_{C}$ is not defined in the lines in $\operatorname{Gr}\left(1, \mathbb{P}^{3}\right)$ that go through the camera center. The readers with a background in Algebraic Geometry will see that this forms a Schubert cell.

The joint camera map for lines is defined as

$$
\begin{align*}
\phi_{\mathcal{C}}: & \operatorname{Gr}\left(1, \mathbb{P}^{3}\right) \longrightarrow\left(\operatorname{Gr}\left(1, \mathbb{P}^{2}\right)\right)^{m} \\
& L \longmapsto\left(v_{C_{1}}(L), \ldots, v_{C_{m}}(L)\right) . \tag{3}
\end{align*}
$$

In the Computer Vision setting, the lines in $\operatorname{Gr}\left(1, \mathbb{P}^{3}\right)$ are considered world lines and the lines in $\operatorname{Gr}\left(1, \mathbb{P}^{2}\right)$ are called image lines. For each world line $L$, the tuple $\phi_{\mathcal{C}}(L)$ is a line correspondence.

For a fixed camera arrangement $\mathcal{C}=\left(C_{1}, \ldots, C_{m}\right)$ the triangulation problem consists precisely in finding elements in the fibers of the joint camera maps, that is, given a point correspondence $\left(x_{1}, \ldots, x_{m}\right)$, its triangulation is a point $X \in \mathbb{P}^{3}$ such that $\varphi_{\mathcal{C}}(X)=\left(x_{1}, \ldots, x_{m}\right)$, and, similarly, given a line correspondence $\left(\ell_{1}, \ldots, \ell_{m}\right)$ its triangulation is a line $L \in \operatorname{Gr}\left(1, \mathbb{P}^{3}\right)$ such that $\phi_{\mathcal{C}}(L)=\left(\ell_{1}, \ldots, \ell_{m}\right)$.


Figure 1: Structure from Motion pipeline. In the final ${ }_{3} D$ model the cameras are in red and the triangulated points are in black. The reconstruction was done with COLMAP [7] using their data set person-hall.

## Multiview varieties

In practice the triangulation is done using noisy data. The noise comes from lens distortion in the cameras, different light effects, or different quality images. This implies that the point or line correspondences obtained in the matching process might not be in the image of the joint camera maps, but are close enough. This is why understanding the images of $\varphi_{\mathcal{C}}$ and $\phi_{\mathcal{C}}$ is necessary. Here is where the multiview varieties come into play.

The Point multiview variety, denoted $\mathcal{M}_{\mathcal{C}}$, is defined as the Zariski closure of $\operatorname{Im}\left(\varphi_{\mathcal{C}}\right)$. Similarly, the Line multiview variety is the Zariski closure of $\operatorname{Im}\left(\phi_{\mathcal{C}}\right)$.

The point and line multiview varieties are, respectively, the smallest algebraic sets containing all the perfect point and line correspondences. They model perfect data. If we understand these varieties, then we can correct the error of a noisy correspondence by triangulating its closest point in the corresponding multiview variety.

The point multiview variety has been thoroughly studied (see for example [3] or [8] for the basic properties). We highlight the fact that a Groebner basis for its vanishing ideal is known [1] and its Euclidean distance degree is known [5]. For this note we just mention the following theorem from [8] that gives a set of polynomials that cut out the point multiview variety.

Theorem 1. Let $\mathcal{C}=\left(C_{1}, \ldots, C_{m}\right)$ be an arrangement of pinhole cameras, and define the $3 m \times(m+4)$ matrix

$$
A(\mathbf{u})=\left(\begin{array}{ccccc}
C_{1} & \mathbf{u}_{1} & 0 & \cdots & 0 \\
C_{2} & 0 & \mathbf{u}_{2} & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
C_{m} & 0 & 0 & \cdots & \mathbf{u}_{\mathbf{m}}
\end{array}\right)
$$

where $\mathbf{u}_{\mathbf{i}}=\left(u_{i 1}, u_{i 2}, u_{i 3}\right)^{T}$ for $i=1, \ldots, m$ are the variables associated with the image of each camera. The multiview variety is cut out by the maximal minors of the matrix $A(\mathbf{u})$ above.

More recently in [2] the authors start the study of the line multiview variety. They introduce the formal definition that we saw above, provide a set of polynomials that cut out the variety, find its singular locus, and compute its multidegree. The results are given for camera arrangements such that at most four of them are collinear, this means that the they are valid for random camera arrangements or, in the language of Algebraic Geometry, the results are valid generically. We highlight the following theorem from [2]

Theorem 2. Given an arrangement of pinhole cameras $\mathcal{C}=$ $\left(C_{1}, \ldots, C_{m}\right)$, denote by $\ell_{i}$ the point in the dual of $\mathbb{P}^{2}$ defining the line $v_{C_{i}}(L)$ in $\operatorname{Gr}\left(1, \mathbb{P}^{2}\right)$. The line multiview variety $\mathcal{L}_{\mathcal{C}}$ is equal to the set

$$
\left\{\left(\ell_{1}, \ldots, \ell_{m}\right) \in\left(\mathbb{P}^{2}\right)^{m} \mid \operatorname{rank}\left(\left[C_{1}^{T} \ell_{1}, \ldots, C_{m}^{T} \ell_{m}\right]\right) \leqslant 2\right\}
$$

if and only if no four cameras are collinear.

From the geometric point of view, the point multiview variety contains the tuples of points such that their back-projected lines intersect in a point (see Figure 2). Similarly, the line multiview variety contains all the line tuples whose back-projected planes intersect either in a plane or a line (see Figure 3).


Figure 2: In red a point correspondence for 3 cameras. The back projected lines of each point are depicted in gray. Since they intersect in the blue point, the point correspondence is in the point multiview variety $\mathcal{M}_{\mathcal{C}}$.


Figure 3: In red a line correspondence for 3 cameras. The back projected planes of the image lines are depicted in gray. Since they intersect in the blue world line, the line correspondence is in the line multiview variety $\mathcal{L}_{\mathcal{C}}$.

## Error correction for real data

In practice, the equations obtained from Theorem 1 and Theorem 2 can be used to develop solvers for triangulation in SfM pipeline. Indeed, triangulating a point cloud requires finding solutions of the same parametric system of polynomial equations as many times as data points. Computer vision engineers have developed implementations that allow for a very fast solution of specific systems coming from theoretical results as the ones introduced above (see for example [ $4,6,7]$ ).
As a final remark we highlight that in [2] the authors conduct numerical experiments to estimate the number of critical points of the distance funcion to the line multiview variety. This is a first approach to the Euclidean Distance degree of the Line multiview variety and it measures the complexity of the triangulation problem using algebraic methods. Although the numerical experiments suggest that this degree is polynomial in the number of cameras, this is still an open question.

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## Research focus

Undecidable trajectories in Euclidean ideal fluids，after［2］， by Robert Cardona ${ }^{\text {『 }}$（UB ${ }^{\text {® }}$ ），Eva Miranda ${ }^{\text {『 }}$（DMAT ${ }^{\text {® }}$ ，IMTech ${ }^{\text {『 }}$ ， CRM $^{\text {® }}$ ），and Daniel Peralta－Salas ${ }^{\text {® }}$（CSIC ${ }^{\text {© }}$ ， ICMAT $\left.^{\text {® }}\right)^{*}$
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## 1．Introduction

Fundamental to the understanding of physical phenomena and dynamical systems in general is the study of the computational complexity that may arise in a given class of systems．This complexity can include undecidable phenomena and compu－ tational intractability，which is relevant not only from a purely theoretical point of view but also in terms of applications to developing algorithms to determine the long－term behavior of a given physical system．

Several dynamical systems have been shown to exhibit un－ decidable trajectories：there exist explicitly computable initial conditions and open sets of phase space for which determining if the trajectory will intersect that open set can be undecidable from an algorithmic point of view．These include ray tracing problems in 3 D optical systems［6］，neural networks［7］，and more recently ideal fluid dynamics $[2,3]$ ，a problem asked in the gos by Moore［5］．In［3］，it was shown that in compact ${ }_{3}$ D domains，one can find examples of stationary ideal fluids that possess undecidable trajectories．The caveat of the proof is that the Riemannian ambient metric is not canonical in any sense，for instance，the proof does not work to construct such examples in the standard flat three－torus，the standard round sphere，or the standard Euclidean space．

In this note we give a short introduction to the ideas de－ veloped in［2］，where we construct stationary ideal fluids in the standard Euclidean space $\mathbb{R}^{3}$ ，i．e．，equipped with the flat metric，that possess undecidable trajectories．The price to pay to obtain solutions in a space with a fixed Riemannian metric like the Euclidean one is working on a non－compact space and obtaining solutions that do not have finite energy．

## 2．Undecidability and Turing machines

The most used technique to prove that a class of dynamical systems might exhibit undecidable trajectories is by construct－ ing an example of that system that is＂Turing complete＂．This means roughly that the system encodes the evolution of any Turing machine，which is a symbolic system encoding a certain algorithm．Let us recall what a Turing machine is．

## Turing machines

A Turing machine is defined as $T=\left(Q, q_{0}, q_{\text {halt }}, \Sigma, \delta\right)$ ，where $Q$ is a finite set（called＂states＂），including an initial state $q_{0}$ and a halting state $q_{\text {halt }}$ ，another finite set $\Sigma$ called the alphabet of symbols and that contains a blank symbol that we denote by a zero，and a transition function

$$
\delta:(Q \times \Sigma) \longrightarrow(Q \times \Sigma \times\{-1,0,1\})
$$

that will encode the dynamics（or＂algorithm＂）．A configuration of the machine at a certain step of the algorithm is given by a
pair in $Q \times \mathcal{A}$ ，where $\mathcal{A}$ denotes the（countable）set of infinite sequences in $\Sigma^{\mathbb{Z}}$ that have all but finitely many symbols equal to 0 ．An＂input＂of the algorithm，or starting configuration of the machine，is given by a pair of the form $\left(q_{0}, t\right)$ ，where $q_{0}$ is the initial state and $t=\left(t_{i}\right)_{i \in \mathbb{Z}}$ is an arbitrary sequence in $\mathcal{A}$ ， which is commonly referred to as the tape of the machine．

The algorithm works as follows．Let $(q, t) \in Q \times \mathcal{A}$ be the configuration at a given step of the algorithm．

1．If the current state is $q_{\text {halt }}$ then halt the algorithm and return $t$ as output．

2．Otherwise，compute $\delta\left(q, t_{0}\right)=\left(q^{\prime}, t_{0}^{\prime}, \varepsilon\right)$ ，where $t_{0}$ de－ notes the symbol in position zero of $t$ ．Let $t^{\prime}$ be the tape obtained by replacing $t_{0}$ with $t_{0}^{\prime}$ in $t$ ，and shifting by $\varepsilon$ （by convention +1 is a left shift and -1 is a right shift）． The new configuration is $\left(q^{\prime}, t^{\prime}\right)$ ，and we can go back to step 1.

We emphasize that there is no loss of generality in the re－ striction to those sequences $\mathcal{A} \subset \Sigma^{\mathbb{Z}}$ that have＂compact sup－ port＂，meaning that all but finitely many symbols are the blank symbol 0 ．The set $\mathcal{P}:=Q \times \mathcal{A}$ is the set of configurations， and the algorithm determines a global transition function

$$
\Delta: Q \backslash\left\{q_{\text {halt }}\right\} \times \mathcal{A} \rightarrow \mathcal{P}
$$

that sends a configuration to the configuration obtained after applying one step of the algorithm．
Let us finish this section with two more facts about Turing machines．When reproducing an algorithm，one would like to know whether a given Turing machine with a given ini－ tial configuration will eventually reach a configuration whose state is the halting state，or if the algorithm will keep running forever．This is known as the halting problem and is known to be（computationally）undecidable as shown by Alan Turing in 1936．This means that there is no algorithm that，given any Turing machine $T$ and any of its initial configurations $c$ ， will answer in finite time whether $T$ halts with $c$ or not．A particular consequence of computational undecidability in this context is that for some pairs $(T, c)$ ，the statement＂$T$ halts with $c^{\prime \prime}$ can be true／false but unprovable，i．e．，undecidable in the sense of Gödel．

The second fact that we will need is that there exist＂uni－ versal Turing machines＂．Those are Turing machines that can simulate in some sense any other Turing machine，and thus that are capable of reproducing any possible algorithm．One can think of those as a＂compiler＂in modern computer sci－ ence．More formally there are different definitions of universal Turing machine，we will use one that is easier to state and that is sufficient for our purposes．

Definition 1．A Turing machine $T_{U}$ is universal if the follow－ ing property holds．Given any other Turing machine $T$ and an initial configuration $c$ of $T$ there exists an initial configuration $c_{U}(T, c)$ of $T_{U}$ ，that depends on $T$ and its initial configuration， such that $T$ halts with $c$ if and only if $T_{U}$ halts with $c_{U}$ ．

[^0]In particular, determining whether a universal Turing machine with a given initial condition will ever halt is a (computationally) undecidable problem, and there exist initial configurations of $T_{U}$ for which it is (logically) undecidable to determine if $T_{U}$ will halt with that initial configuration.

## Turing complete systems

Having understood the relation between Turing machines and undecidability, we relate them to dynamical systems in the following way. Let $X$ be a dynamical system on a topological phase space $M$, where $X$ can be either discrete or continuous on a finite or infinite-dimensional phase space. For concreteness, we can keep in mind the example of an autonomous flow on a smooth manifold.

Definition 2. A dynamical system $X$ on $M$ is Turing complete if there exists a universal Turing machine $T_{U}$ such that for each initial configuration $c$ of $T_{U}$, there exists a (computable) point $p_{c} \in M$ and a (computable) open set $U_{c} \subset M$ such that $T_{U}$ halts with input $c$ if and only if the positive trajectory of $X$ through $p$ intersects $U_{c}$.

In this case, the halting of a given configuration can be deduced from the evolution of an orbit of $X$. It is essential to require that $p$ and $U_{c}$ are in some sense explicit, namely computable, since otherwise, one could run into trivial systems being Turing complete. If we are working on a manifold, a point $p$ is computable (in terms of $c$ ) if in some chart the coordinates of $p$ can be exactly computed in finite time (in terms of $c$ ), for instance having explicit rational coordinates. Computability of an open set $U_{c}$ can be loosely defined as saying that one can explicitly approximate $U_{c}$ with any given precision. This notion is formalized in a subject called computable analysis. A Turing complete system has undecidable trajectories, meaning that there exist an explicit point $p$ and open set $U$ for which determining if the trajectory of $p$ reaches $U$ is an undecidable statement. This is different from being chaotic, where the sensitivity to initial conditions yields a practical unpredictability of trajectories since we are saying that even if we know exactly the initial point $p$, the long-term behavior can be completely unpredictable.

In practice, most Turing complete systems are constructed in the following way. We first encode, in a computable way, each configuration $(q, t)$ of a universal Turing machine $T_{U}$ as a point or an open set $U_{(q, t)}$ of the phase space $M$. For our purposes, assume we encode the initial configuration $\left(q_{0}, t^{i n}\right)$ as points $p_{\left(q_{0}, t^{i n}\right)}$, and every other configuration as an open set. We then require that for each initial configuration $\left(q_{0}, t^{i n}\right)$, the trajectory of $X$ through $p_{\left(q_{0}, t^{i n}\right)}$ sequentially intersects the sets corresponding to the configurations obtained by iterating the Turing machine starting with $\left(q_{0}, t^{i n}\right)$. Namely, the trajectory through $\left(q_{0}, t^{i n}\right)$ will first intersect the set that encodes $\Delta\left(q_{0}, t^{i n}\right)$, then the set that encodes $\Delta^{2}\left(q_{0}, t^{i n}\right)$ and so on, without intersecting any other coding set in between. With this property, we can consider the open set $U$ obtained as the union of all the open sets $U_{\left(q_{\text {halt }}, t\right)}$ for each $t \in \mathcal{A}$, and the trajectory through $p_{\left(q_{0}, t^{i n}\right)}$ will intersect $U$ if and only if the machine $T_{U}$ halts with initial configuration $\left(q_{0}, t^{i n}\right)$.

## 3. Constructing stationary ideal fluids that are Turing complete

Having defined Turing complete systems, let us now describe the equations for which we would like to construct a solution that is Turing complete.

## The Euler equations and sketch of the main theorem

The motion of an incompressible fluid flow without viscosity is modeled by the Euler equations. In $\mathbb{R}^{3}$, the equations can be
written as

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} X+\nabla_{X} X \quad=-\nabla p \\
\operatorname{div} X=0
\end{array}\right.
$$

where $p$ stands for the hydrodynamic pressure and $X$ is the velocity field of the fluid (a non-autonomous vector field). Here $\nabla_{X} X$ denotes the covariant derivative of $X$ along $X$. If $X$ is a stationary solution, i.e., time independent, then the first equation is equivalent to $X \times \operatorname{curl}(X)=\nabla B$, with $B:=p+\frac{1}{2}\|X\|^{2}$ and curl denotes the standard curl operator induced by the Euclidean metric. A vector field that satisfies $\operatorname{curl}(X)=\lambda X$ for some constant $\lambda \neq 0$ is called a Beltrami field. It is a particular case of a stationary Euler flow with constant Bernoulli function. The main theorem we proved in [2, Theorem 1] is:
Theorem. There exists a Turing complete Beltrami field $u$ in Euclidean space $\mathbb{R}^{3}$.
The strategy of the proof can be sketched as follows.
(1) We show that there exists a Turing complete vector field $X$ in the plane $\mathbb{R}^{2}$ that is of the form $X=\nabla f$ where $f$ is a smooth function.
(2) Furthermore, we require that if we perturb $X$ by an error function $\varepsilon: \mathbb{R}^{2} \rightarrow \mathbb{R}$ that decays rapidly enough at infinity, then we obtain a vector field that is Turing complete as well.
(3) We show that a vector field in $\mathbb{R}^{2}$ of the form $X=\nabla g$, where $g$ is an entire function, can be extended to a Beltrami field $v$ in $\mathbb{R}^{3}$ such that $\left.v\right|_{z=0}=X$. That is $v$ leaves the plane $\{z=0\}$ invariant and coincides with $X$ there.
(4) We approximate $f$ by an entire function $F$ with an error that decays rapidly enough. Hence $\tilde{X}=\nabla F$ is Turing complete and extends as a Beltrami field $u$ on $\mathbb{R}^{3}$. It easily follows that $u$ is Turing complete as well.

In this note, we will sketch the arguments of steps (1) and (2). The third step is done via a global Cauchy-Kovalevskaya theorem adapted to the curl operator. The fourth step is a general result about approximation of smooth functions by entire functions with errors with arbitrary decay [4].
Weakly robust Turing complete gradient flow in the plane The goal of this section is to construct a Turing complete gradient flow on $\mathbb{R}^{2}$ and sketch how to make sure that its Turing completeness is robust under perturbations that decay fast enough at infinity. Following the recipe explained in Section 2, we will first show how to encode the configurations and initial configurations of a given universal Turing machine $T_{U}$ into $\mathbb{R}^{2}$. Without loss of generality, we assume that $T_{U}=\left(Q, \Sigma, q_{0}, q_{\text {halt }}, \delta\right)$ with $Q=\{1, \ldots, m\}$ for some $m \in \mathbb{N}$ and $\Sigma=\{0,1\}$.

The encoding. We first construct an injective map from $\mathcal{P}=Q \times \mathcal{A}$ to $I=[0,1]$, where we recall that $\mathcal{A}$ is the set of sequences in $\Sigma^{\mathbb{Z}}$ with finitely many non-zero symbols. Given $(q, t) \in \mathcal{P}$, write the tape as

$$
\cdots 000 t_{-a} \cdots t_{b} 00 \cdots
$$

where $t_{-a}$ is the first negative position such that $t_{-a}=1$ and $t_{b}$ is the last positive position such that $t_{b}=1$. If every symbol in a negative (or positive) position is zero, we choose $a=0$ (or $b=0$ respectively). Set the non-negative integers given by concatenating the digits $s:=t_{-a} \cdots t_{-1}, r:=t_{b} \cdots t_{0}$, and introduce the map

$$
\varphi(q, t):=\frac{1}{2^{q 3^{r} 5^{s}}} \in(0,1)
$$

which is injective and its image accumulates at 0 . There exist pairwise disjoint intervals $I_{(q, t)}$ centered at $\varphi(q, t)$, for instance of size $\frac{1}{4} \varphi(q, t)^{2}$. To introduce an encoding into $\mathbb{R}^{2}$ we proceed as follows. Fix $\epsilon>0$ small, we encode ( $q, t$ ) as

$$
U_{(q, t)}:=\bigcup_{j, k=0}^{\infty} I_{(q, t)}^{j} \times(k-\epsilon / 2, k+\epsilon / 2)
$$

where $I_{(q, t)}^{j}:=I_{(q, t)}+(2 j, 2 j)$. In other words, we are looking at any interval of the form $I^{j, k}:=[2 j, 2 j+1] \times\{k\} \subset \mathbb{R}^{2}$, with $j, k \in \mathbb{N}$, and considering an $\varepsilon$-thickening of $I_{(q, t)}$ understood as a subset of $I^{j, k} \cong[0,1]$. Figure 1 gives a visualization of part of one of the open sets $U_{(q, t)}$ in a region of the plane.


Figure 1. The open set $U_{(q, t)}$ is the $\varepsilon$-thickening of the intervals $I_{(q, t)}^{j} \times\{k\}$ The countable set of initial configurations $\mathcal{P}_{0}=\left\{\left(q_{0}, t\right) \mid t \in\right.$ $\mathcal{A}\}$ admits a (computable) ordering which we will not specify, so that we can write it as $\mathcal{P}_{0}=\left\{c_{i}=\left(q_{0}, t^{i}\right) \mid i \in \mathbb{N}\right\}$. Given $c_{i}$, the initial condition associated to the vector field that we will construct will be $p_{c_{i}}=\left(\varphi\left(c_{i}\right)+i, 0\right) \in \mathbb{R}^{2}$. This corresponds to the point $\varphi\left(q_{0}, t^{i}\right)$ of the copy $I^{i, 0}$ of the several intervals we considered.
Integral curves capturing the steps of the algorithm. Iterating the global transition function from an initial configuration $c_{i}=\left(q_{0}, t^{i}\right)$ gives a countable sequence of configurations $c_{i}^{k}=\left(q_{k}, t_{k}^{i}\right)=\Delta^{k}\left(c_{i}\right)$ for each $k$ an integer greater than 1. On each band $[2 i, 2 i+1] \times[0, \infty)$, we construct a smooth curve $\gamma_{i}$ such that $\gamma_{i} \cap\{[2 i, 2 i+1] \times\{k\}\}$ is the point $\left(2 i+\varphi\left(q_{k}, t_{k}^{i}\right), k\right)$, which lies in $U_{\left(q_{k}, t_{k}^{i}\right)}$, see Figure 2.


Figure 2. Integral curves following the computations of the Turing machine
We conclude by constructing a gradient field $X=\nabla f$ such that each $\gamma_{i}$ is an integral curve of $X$. Observe now that
given an initial condition $c_{i}$, the integral curve through $p_{c_{i}}$ will intersect sequentially the open sets $U_{\left(q_{k}, t_{k}^{2}\right)}$, thus keeping track of the computations of the machine with initial configuration $c_{i}$. One easily shows that $X$ is Turing complete, where to each $c_{i}$ we assign the initial condition $p_{c_{i}}$, and the open set $U$ for which the trajectory through $c_{i}$ intersects $U$ if and only if the machine halts with initial configuration $c_{i}$ is simply $U=\bigcup_{t \in \mathcal{A}} U_{\left(q_{\text {halt }}, t\right)}$, that is, every open set encoding a halting configuration.

Weak robustness and conclusion. Recall that in order to apply the Cauchy-Kovalevskaya theorem, we need $X$ to be the gradient of an entire function. To achieve this, we construct $X$ in a way that the flow normally contracts towards each curve $\gamma_{i}$ at a strong enough rate. This can be used to show that if we perturb $X$ by an error function $\varepsilon(x, y)$ with fast decay at infinity, we obtain a vector field that is again Turing complete. This is because even if the curve $\gamma_{i}$ will no longer be an integral curve, the integral curve through any of the points $p_{c_{i}}$ of the perturbed vector field will still intersect sequentially the open sets $U_{\left(q_{k}, t_{k}^{i}\right)}$, hence capturing the computations of the Turing machine. The fast decay of the error is necessary since the open sets $U_{\left(q_{k}, t_{k}^{i}\right)}$ have no uniform lower bound on their size. This is because the intervals $I_{(q, t)}$ accumulate at zero, and hence their size tends to zero. One can estimate the decay rate of the size of the open sets that need to be intersected by the curves in terms of the distance to the origin, and hence robustness can be achieved under fast decay errors. The construction concludes by approximating $f$ by an entire function $\tilde{f}$ (using [4]), and applying the Cauchy-Kovalevskaya theorem for the curl to the Cauchy datum $\nabla \tilde{f}$ on the plane $\{z=0\}$.

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## Chronicles

Why do we aspire to be second－rate mathematicians when we can be first－rate scientists？
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Received on 19 September， 2023.


On September 14，2023，professor G．Geoffrey Vining ${ }^{\text {® }}$ gave the opening talk of the George Edward Pelham Box school year at the FME $^{\circlearrowright}$ ．The FME has dedicated the 2023－24 school year to G．E．P．Box ${ }^{\text {® }}$（1919－2013），of whom G．G．Vining was a former student．

The conference opened with an address by the Dean of the FME，professor Jordi Guàrdia，${ }^{(1)}$ and continued with the intro－ duction of the speaker by Xavier Tort－Martorell ${ }^{\text {T }}$ ．

The title of professor Vining＇s lecture was＂First－Rate Scien－ tist or Second－Rate Mathematician＂，which was inspired by the following quote from professor G．P．Box：＂Why do we aspire to be second－rate mathematicians when we can be first－rate scientists？＂

The lecture began by explaining that G．P．Box，a chemist by training，was a sergeant during the Second World War at a research station，in which the consequences of a chemical war were investigated with animals．As a result of the research he was carrying out，he contacted Sir R．A．Fisher ${ }^{〔}$ ，${ }^{(2)}$ known for being the father of the design of experiments，and later ended up marrying one of his daughters．Almost always，G．P．Box＇s contributions arose from real problems that appeared as a re－ sult of his dedication to improving the quality and productivity
of industrial processes．This is how his seminal contributions to the design of experiments and prediction through time series emerged among the most well－known．He was also famous for his quotes，like the one that inspires the lecture we＇re talking about．
The central part of the lecture versed on the explanation of two NASA projects with which professor Vining has been involved．The first concerned the study of the reliability of carbon－lined vessels through which gas passes at very high pressure，and the second the design of jet turbine engines．

Throughout the lecture the importance of the design of ex－ periments while doing research was highlighted，as well as the need to verify that the assumptions made while defining the models that will be assumed throughout the different analyzes are really acceptable in the environment in which the research is carried out．The speaker ended the lecture by emphasizing the importance of，regardless of our background，considering ourselves scientists in the broadest sense．Research is cur－ rently multidisciplinary and requires skills to interact produc－ tively with other researchers with very diverse backgrounds．
When asked by a floor participant if he had any sugges－ tions to improve communication between statisticians and re－ searchers，professor Vining mentioned the importance of having scientists who act as a bridge and who have the ability to take decisions．

（1）The NL interviewed him recently：NLo4 ${ }^{\complement}$ ，pp．6－8．
（2）He was dedicated the FME 2012－13 term ${ }^{『}$

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