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IMTECH 5*a*

Newsletter

Issue 5, Advances 5*a*, Feb 2023

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Editorial

IMTech/NL Advances

Starting with the NL05, there will be, whenever possible, monthly Advances to provide an earlier access to materials. The Advances of a given issue will be collected in the full issue published at the end of the current semester. Thus 5*a* features Advances of NL05 corresponding to February 2023, and NL05 will be published in July 2023.

NL05*a* includes interviews with Xavier Ros-Oton and with Olli Saari, the research note *Quantum character varieties*, by Marta Mazzocco, and a review of Joaquim Bruna's recent book *Analysis in Euclidean Spaces*.

Events and News

Eva Miranda will deliver the Hardy lectures 2023, starting at the end on May in Cambridge. For further details, see her "Hardy Tour" Web page.

We also celebrate that Ingrid Deaubechies has been awarded the Wolf Prize in Mathematics.

Acknowledgments

We are thankful to Juan Luis Vázquez for having shared his presentation of Xavier Ros-Oton to the Real Academia de Ciencias on 11 January 2023. The IMTech/NL has used it as a basis for the bio of Ros-Oton at the beginning of his interview.

Xavier Ros-Oton delivered a lecture on 11 January 2023, on the occasion of receiving the diploma of correspondent member of Real Academia de Ciencias (see the picture at the end), with the title *Teoría de regularidad para Ecuaciones en Derivadas Parciales elípticas* (Regularity Theory for Elliptic Partial Differential Equations). This is currently the last distinction in recognition of his research contributions on a wide variety of topics in the field of partial differential equations (PDE), which are "the equations that move the world", as he himself asserted in one of his lectures. These equations play a central role in all areas of physics, as well as engineering, biology and finance, and also in some of the most famous problems in pure mathematics.

Born in Barcelona in April 1988, he completed his Bachelor's Degree in Mathematics at the Polytechnic University of Catalonia (UPC) in 2010, the master thesis in 2011, and his doctoral thesis in June 2014, both under the supervision of Xavier Cabré. Consequence of his thesis was the remarkable article [1], co-authored with Joaquim Serra (interviewed in NL02, pp. 5-6), which is the most cited article of the year 2014 in MathSciNet (among all mathematics articles) and as of today has over 730 citations (according to Google Scholar).

Ros-Oton pursued his postdoctoral studies at the University of Texas at Austin) from 08/2014 to 08/2017, where he collaborated with the great teacher Luis Caffarelli, world leader in problems of free boundaries, and with Alessio Figalli, young star of the Italian PDE. In 2017 he was called to the University of Zurich as Assistant Professor, and there he continued his close collaboration with Alessio Figalli, who in 2018 was awarded the Fields Medal at the ICM-2018. In 2020 he returned to Spain after being appointed as Full Professor at the UB and as ICREA Professor.

The results of his research have been published in the best mathematical journals worldwide and it is said that he is currently the most cited mathematician of his age in the world. In the few years after his PhD, he has been awarded various distinctions: the Vicent Caselles Prize in 2015 (awarded by the RSME and the FBBVA); the Rubio de Francia Prize in 2017 (awarded by the RSME); the Antonio Valle Prize to Young Researchers in 2017 (awarded by SeMA); the Princess of Girona Foundation Scientific Research Award in 2019; and the Stampacchia Gold Medal in 2021. The mention of the Princess of Girona Foundation Award says: *For being one of the most brilliant mathematicians with the greatest impact worldwide in his age group. His research on partial differential equations, a central field in many scientific disciplines, has had a very prominent impact on the scientific community.*

In 2018, he received an ERC Starting Grant. In 2019 he was one of the plenary speakers at the RSME biennial congress. A person who loves communication and an excellent speaker, he has increasingly participated in these years in congresses, seminars and mini-courses in various countries, has written popular articles and has collaborated with mathematical societies (Spanish and Catalan). The book [2] (with Xavier Fernández-Real) has just appeared in the Zurich Lectures in Advanced Mathematics collection of the EMS.

NL. ***Let us begin at the beginning: When did you decide to become a mathematician? Did you consider other options? What were the main factors that tipped your resolution?***

I decided to study mathematics during the last two years of high school. A key factor that fostered my interest in mathematics were the mathematical competitions organized by the SCM (such as the Cangur and math olympiads), and especially the preparation courses of Prof. Grané at the FME. After participating in these activities during one or two years, it became very clear to me that I wanted to become a mathematician, and pursue a research career in mathematics.

Which memories do you have as a student at UPC and how did your training at UPC influence your academic career?

I have very fond memories of my time as a student at the FME: I had a really good time and I learned a lot; I'd definitely make the same choice again. During my last year as an undergraduate student I had to decide the topic and advisor of my PhD, and once I decided that I was mostly interested in Analysis I was very lucky to have Xavier Cabré at UPC.

You did your PhD at UPC under the supervision of Xavier Cabré. Which are your recollections as a graduate student?

Thanks to Xavier Cabré, I learned a lot during my PhD, and worked on a variety of mathematical problems. I not only learned a lot directly from him, but also participated in many conferences and summer schools, which allowed me to interact with several top mathematicians from Europe and US. Furthermore, it was also crucial for me to have Joaquim Serra working on his PhD at the same time: he is still one of my main collaborators and I learned a lot from him, too.

After your PhD you went to Austin and then to Zürich. How was your experience there in personal and professional terms? Which were the main research outcomes from that time?

During my time as a postdoc in Austin I started working on a new topic: free boundary problems. Such a change allowed me to learn a different area of research, while at the same time exporting the knowledge gained during my PhD to a different context. I was lucky enough to collaborate with A. Figalli and L. Caffarelli, and obtained several important results with them. These are the main results for which I received prizes from the RSME, SeMA, and the Fundación Princesa de Girona.

After three years as a postdoc in Austin, I moved to Universität Zürich, where I could continue working with A. Figalli and J. Serra. It was a great time, both from the personal and professional point of view, and it was in Zürich where I obtained my best mathematical results.

Could you please tell us about your own research and the problems that most interest you?

I have worked on quite diverse topics, all related to the broad areas of PDE and Calculus of Variations. The main topic on which I have worked in the last years is one of the most basic and important question in PDE theory: to understand whether all solutions to a given PDE are smooth or if, instead, they may have singularities. Some of my main contributions have been in the context of free boundary problems, that is, PDE problems that involve unknown interfaces. From the mathematical point of view, they give rise to extremely challenging questions, and their study is closely connected to geometric measure theory. In particular, the study of free boundary problems has a strong geometrical flavor.

What problems in areas of your interest would you like to see solved in the coming years?

A type of question I like very much (and for which very little is known) is to understand "generic regularity" for PDE problems. For example, even if some solutions may develop singularities, is it possible to prove that is it an extremely unlikely scenario? In other words, can one prove that in some PDE problems singularities appear with probability zero? I would like to see many more developments in this direction.

How do you value the prizes and distinctions you have been awarded in recognition of the quality of your research?

I feel very honored and lucky to have received these distinctions, and I know there are other mathematicians who would deserve them, too. Moreover, they are awarded for works that I have established with many collaborators, so this is a recognition for them, too.

You have since 2020 a research position in Barcelona. How do you see the research system in Spain and what do you think are its strengths and weaknesses?

Unfortunately, I think that we are still quite far from the research systems and environments of more advanced countries. The main strength is that we have several mathematicians who are working here and doing an excellent job. The main weakness I see is the bureaucracy and lack of administrative support that researchers have, compared to what I saw in the US, Switzerland, and other countries I visited.

References

[1] Xavier Ros-Oton and Joaquim Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, Journal de Mathématiques Pures et Appliquées **101** (2014), no. 3, 275-302.

[2] Xavier Fernández-Real and Xavier Ros-Oton, *Regularity theory for elliptic PDE*, EMS, 2022.

Olli Saari started with a Ramón y Cajal contract at the **UPC** in January 2023. His research is focused on harmonic analysis and analysis of partial differential equations. He completed his master's degree in engineering physics and mathematics at Aalto University in 2013 and continued then as a doctoral student in the same institution under supervision of Juha Kinnunen, funded by the Vilho, Yrjö and Kalle Väisälä Fund. During his doctoral studies, he spent a semester as a visiting researcher at UPV/EHU in Bilbao. After finishing his PhD in 2016, he spent the fall semester 2016 at the University of Bonn and the spring semester 2017 in the Mathematical Sciences Research Institute (California) as a post-doctoral fellow. Starting from the fall 2017 and ending with the year 2022, he was employed by the University of Bonn as a post-doctoral assistant in the working group of Christoph Thiele, after which he moved to Barcelona to his current position.

NL. ***Welcome to Barcelona and the UPC, and congratulations for the Ramón y Cajal contract. Could you comment on what features of Barcelona and its mathematical community contributed to your decision to land here with your RyC?***

A Ramón y Cajal contract offers a lot of freedom so the choice was a difficult one. But Barcelona was my clear favourite in the end. The analysis group at UAB is famous and extremely strong. The PDE group at UB is also very strong and was also an important attractive factor. Our working group at the UPC is slightly smaller, but we share a seminar with the other two, so there is regular communication. This is the picture of the mathematical analysis community in Barcelona that I got after discussing with Xavier Cabré about possibility of coming here. The UPC then felt like the university to which I would fit in best. Also beyond mathematics, Barcelona is a very interesting place. I had not really spent much time here before moving in, just a few days for conference trips, but so far I have been very happy.

Your master's degree at Aalto University suggests a strong interdisciplinary mix. How did that atmosphere impact on your professional education? Was it decisive in your research orientation? What was the topic of your master thesis?

The mission of my degree programme at Aalto University was to educate engineers with solid background in physics and mathematics to later move to real world applications in industry. This did not quite happen with me, and in retrospective I even feel I wasted a part of the opportunities that the interdisciplinary tendency around me could have brought. I ended up working on topics rather far from applications, the title of my master's thesis was *Local to global results for John-Nirenberg inequalities*, but I think it was very beneficial to study in an environment with good balance of pure and applied mathematics. I studied courses in numerics, physics and mechanics that I would not have taken elsewhere. But I can't say my research orientation were affected very much by that.

What was the topic of your PhD thesis? What are your recollections of that research period? How was the guidance by, and collaboration with your advisor, Juha Kinnunen?

The title of my PhD thesis was *Weights arising from parabolic partial differential equations*. A good part of mathematical analysis can be traced back to the study of harmonic functions, the solutions of the Laplace equation. One can go back to the basics and replace the stationary potential equation by an evolutionary diffusion equation and try to understand how such a change should be seen by inequalities in analysis that do not have anything to do with partial differential equations at the first sight, although they stem from the study of harmonic functions, in fact. My project was to apply this point of view to what is known as weighted norm inequalities.

The time of writing the PhD thesis was a very special period of my research career. There was a clear unique choice what to work on, as there was an advisor to tell me that, and except for the meetings with my advisor, I was mostly working alone in the beginning. Then the working routines changed at some point. Working extended periods on the same problem without too much external input was certainly a good learning experience, and it helped to build a solid understanding with no details omitted. The research as it started occurring later is more collaborative and also much more fast-paced.

Kinnunen was a very encouraging character for me as an advisor. He was actually not only my PhD advisor but also the advisor for my master's and bachelor's theses so I started working with him even before choosing mathematics as the major subject.

Can you tell us about your experiences as a visiting researcher for four months at UPV/EHU?

I was invited to Bilbao by Ioannis Parissis who had moved there after being a post-doc at Aalto University, and Carlos Pérez. At the time of staying in Bilbao I had already done the research for my thesis and was writing the final form of the thesis. I had therefore time to learn something new. The host group was focused on harmonic analysis whereas my home group at Aalto had been more of a PDE group. My thesis topic was something in between, so I found this change of environment extremely inspiring.

Let us move forward to your postdoctoral journey in Europe and the United States. What aspects of those stays would you like to highlight?

My first two short visits, Bonn and MSRI, were purely for research. During my later and longer stay at Bonn I was also involved in teaching. The research program in MSRI was of course an important experience and opportunity to get to know lots of colleagues and to get an idea about a wide range of topics that they were working on. It also showed clearly that one cannot stay on top of every currently evolving area in research that one understands to some extent but one has to focus a lot more. Researchwise, the most beneficial side of my postdoctoral positions has been the opportunity to widen one's mathematical perspective by getting to know new people and their research. Working with Pascal Auscher in MSRI or Christoph Thiele in Bonn has had a huge impact on how I see mathematics nowadays, probably even more than what I learned during my PhD thesis, not to mention the collaborations with younger colleagues, such as Simon Bortz and Moritz Egert whom I met in MSRI. It has also been interesting to see how differently things work in different countries. There are many ways to arrange research and teaching. Bonn was very different from Finland and the UPC must be another story again.

What vision do you have for your research during the RyC contract? What would you like to achieve? Do you have in mind collaborating with members of the mathematical community, or, more specifically, with participants of the Barcelona Analysis Seminar?

My current contract gives a planning security for a long time, which is an excellent thing to have. I have a few topics on the waiting list that I always wanted to look at more thoroughly when having more time: divergence equation on non-cylindrical domains, regularity of vertical maximal functions and convex sets as bilinear Fourier multipliers. But in reality, what I will end up doing also depends on the people around me. There are the collaborators far away with whom I am in a more or less regular correspondence, and I am also talking to people around here in Barcelona, for instance Albert Mas and Xavier Cabré at UPC. I chose to come here mostly because of the math community so I should try to take advantage of it.

In the mathematical ecosystem of the UPC, there is a wide room for teaching subjects in several studies offered by the FME and CFIS at various levels. Can you share how do you see your role in that respect for the next few years?

Well yes, this is a very nice question to answer. I heard, from several people, that year after year UPC manages to attract very strong students. On the other hand, there are not so many researchers in harmonic analysis involved in the teaching at the moment. There would be use for more people with my background. Going back to the first questions of this interview, choosing UPC instead of one of the other universities in Barcelona has a lot to do with this. I am currently teaching exercises for the advanced PDE in the master's program and supervising two master's theses. I will also be part of the teaching plans of our working group in the future. There will be more students with interest in analysis in the years to come, no doubt, and it will be good for them that I will be here. I don't have a good estimate on the number of students around here yet, but maybe there will even be a chance of introducing a course in harmonic analysis at some point in the future, who knows.

We hope that you will enjoy your contract and that it will turn out to be a most positive time for your academic career. We also wish you good luck!

Thank you!

Research focus

Quantum character varieties

by Marta Mazzocco (School of Mathematics, Birmingham).
Received on 8/2/2023.

In this note we will introduce the concept of character variety in the simple example of a torus with one disk removed. We will show how this is a surface in $\mathbb{C}^3$ defined by a cubic polynomial, called the *Markov cubic*. We will show the relation between the Markov cubic, singularity theory, Painlevé differential equations, and introduce a cluster algebra structure on it which is related to Markov numbers. We will discuss quantisation of the Markov cubic in the context of a wider class of surfaces and relate it to Sklyanin algebra.

Main motivation

"Symmetry, as wide or narrow as you may define its meaning, is one idea by which humans through the ages have tried to comprehend and create order, beauty and perfection" (Hermann Weyl). As mathematicians, we chase symmetry. The most profound and far reaching idea in physics is Emmy Noether theorem: the symmetries of a system imply the existence of conserved quantities along the evolution of that system.

One of the most puzzling symmetries discovered nowadays is mirror symmetry (MS). In string theory, particles are replaced by strings and six extra small dimensions are needed to describe the universe. These are wrapped up in Calabi-Yau (CY) varieties that occur in pairs: CY in the same pair produce equivalent physical theories. Following Noether's idea, one way to comprehend MS is to study its fixed points, namely self mirrors. In this note I will discuss some examples of self mirrors which appear in several different contexts in mathematics.

Toy introduction to MS: Quantum mechanics analogy

Schrödinger picture	*Heisenberg picture*
State vectors evolve in time, operators mostly constant: $\lvert\psi(t)\rangle = U(t,t_0)\,\lvert\psi(t_0)\rangle$. Quantisation of the phase space: *geometric quantisation.*	State vectors are time independent and the observables evolve in time. This is the quantisation of the algebra of functions on the phase space: *deformation quantization.*

By the Stone–von Neumann theorem, these two pictures are equivalent (just a basis change in the Hilbert space).

Mirror symmetry

Similarly to the case of quantum mechanics, in MS we have two different sides: an A-side and a B-side.

Mirror pairs of CY varieties (compact Kähler manifolds with vanishing first Chern class and Ricci flat metric) are of very different nature but the symplectic geometry of the A-side is reflected in the algebraic geometry of its mirror:

A-side	*B-side*
Gromov-Witten (GW) invariants: two objects with the same complex structure have the same GW invariants. (Y,D) log symplectic CY *Strominger-Yau-Zaslow quantisation.*	Landau-Ginzburg models: two objects with the same symplectic structure are equivalent. Spec(Ring) *Deformation quantization (for example Etingof-Ginzburg).*

The physical theories produced by the two sides are equivalent.

The aim of this note is to present a class of examples (affine del Pezzo varieties) that are self mirrors (can be on both sides). These examples are relevant in several branches of mathematics:

- ❏ Number theory
- ❏ Singularity theory
- ❏ Orthogonal polynomials
- ❏ Moduli spaces
- ❏ Painlevé equations
- ❏ Cluster algebras

The Markov cubic

$$\mathcal{M} = \{(x_1,x_2,x_3)\in\mathbb{C}^3 \mid x_1^2+x_2^2+x_3^2-3x_1x_2x_3=0\}$$

$\mathcal{M}\cap\mathbb{Z}^3_+ = \{\text{Markov triples}\}$. Startig from $(1,1,1)$, all Markov triples are produced by permutations and Vieta jumping equivalent to the following three operations called "mutations":

$$\mu_1 : (x_1,x_2,x_3) \to \left(\tfrac{x_2^2+x_3^2}{x_1}, x_2, x_3\right)$$
$$\mu_2 : (x_1,x_2,x_3) \to \left(x_1, \tfrac{x_1^2+x_3^2}{x_2}, x_3\right)$$
$$\mu_3 : (x_1,x_2,x_3) \to \left(x_1, x_2, \tfrac{x_1^2+x_2^2}{x_3}\right)$$

Examples:

$$(1,1,1) \xrightarrow{\mu_2} (1,2,1) \xrightarrow{\mu_1} (5,2,1) \xrightarrow{\mu_2} (5,13,1)$$
$$\xrightarrow{\mu_3} (5,13,194) \xrightarrow{\mu_2} (5,2897,194)$$
$$\xrightarrow{\mu_1} (1686049,2897,194)$$
$$\xrightarrow{\mu_2} (1686049,981277621,194)$$

Note that despite dividing by an integer at each step, the result is not a rational number, but an integer again. This is a consequence of the so-called Laurent phenomenon in cluster algebra.

Cluster algebras

This is a class of commutative rings introduced by Fomin-Zelevinsky in 2002. They are integral domains with some sets of size n called *clusters.* Given a cluster $(x_1,\dots,x_n)$, all other clusters are generated by mutations of the form

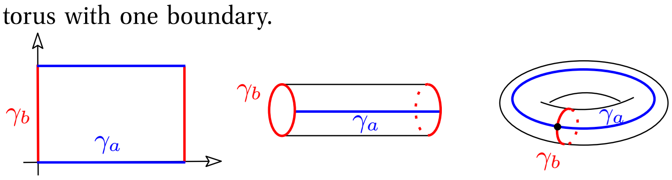
$$\mu_k : x_i \mapsto x_i \ \textbf{for}\ i\neq k, \quad x_k \mapsto x_k' := \tfrac{m_1+m_2}{x_k},$$

where m_1 and m_2 are monomials in $(x_1,\dots,x_n)$ obeying some rules.

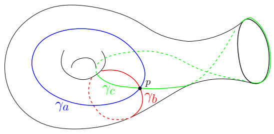
Theorem (Laurent phenomenon). Given any two clusters in a cluster algebra, $(x_1,\dots,x_n)$ and $(\tilde{x}_1,\dots,\tilde{x}_n)$, then $\tilde{x}_i$ is a Laurent polynomial of $(x_1,\dots,x_n)$, $i=1,\dots,n$.

Example. The integers $\mathbb{Z}$ with the Markov triples form a cluster algebra, therefore any component of a Markov triple is a Laurent polynomial of $(1,1,1)$.

The Markov cubic is related to the character variety of a torus with one boundary.



Standard torus $\mathbb{T}^2$: $\pi_1(\mathbb{T}^2,p) = \langle\gamma_a,\gamma_b\rangle \simeq \mathbb{Z}\times\mathbb{Z}$, as $\gamma_a\gamma_b \sim \gamma_b\gamma_a$.



Torus with one boundary:
$\pi_1 = \langle [\gamma_a], [\gamma_b], [\gamma_c] \rangle$, $\gamma_a\gamma_b\gamma_c \sim \gamma_0$,
γ_0 loop around the removed disk.

Its character variety is defined as $\mathrm{Hom}(\pi_1, \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$. To write this as an affine surface, $\forall \gamma \in \pi_1$, $M_\gamma \in \mathrm{SL}_2(\mathbb{C})$, $M_a M_b M_c = M_0$ up to a global conjugation. Thus $\mathrm{tr}(M)$ are invariant and $x_i = \mathrm{tr}(M_i)$ for $i = a, b, c$ satisfy a cubic equation:

$$x_a x_b x_c - (x_a^2 + x_b^2 + x_c^2) = \text{constant}.$$

This is equivalent to the Markov cubic up to a rescaling of x_a, x_b, x_c.

Character variety of the 4-holed sphere

By a quadratic transformation, the Markov cubic is related to the character variety of a sphere with 4 punctures. More generarlly, for the case of 4 holes, this is:

$$\{(x_1, x_2, x_3) \in \mathbb{C}^3 | \\ x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2 = \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4\},$$

where $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{C}$ are parameters.
–Monodromy manifold of the sixth Painlevé equation.
–Versal deformation of a du Val D_4 singularity at $(2, 2, 2)$.
–Most importantly, it is self mirror (see [1]).

I will show how to quantise a wide class of affine surfaces that contains this example as a sub-case.

Quantization of affine surfaces

We focus on a special class of affine surfaces of the form

$$\mathcal{M}_\varphi = \{(x_1, x_2, x_3) \in \mathbb{C}^3 | \varphi(x_1, x_2, x_3) = 0\},$$

where

$$\varphi(x_1, x_2, x_3) = x_1x_2x_3 + \phi_1(x_1) + \phi_2(x_2) + \phi_3(x_3),$$

with $\deg(\phi_i) \leqslant 6$. Examples:
– Affine del Pezzo surfaces: blowup of $9 - d$ points of $\mathbb{P}^2$.
– Deformations of elliptic singularities (e.g. affine cone surfaces with elliptic singularity) [2] and Kleinian sigularities [3] (weighted projective del Pezzo with a nodal singularity).

Poisson structure. Given any $\varphi \in \mathbb{C}[x_1, x_2, x_3]$, we get a Poisson bracket on $\mathbb{C}[x_1, x_2, x_3]$, i.e. a bilinear map

$$\{\cdot, \cdot\} : \mathbb{C}[x_1, x_2, x_3] \times \mathbb{C}[x_1, x_2, x_3] \to \mathbb{C}[x_1, x_2, x_3]$$

that is skew-symmetric and satisfies the Leibniz rule and the Jacobi identity. This is defined in terms of the potential φ as:

$$\{x_1, x_2\} = \tfrac{\partial \varphi}{\partial x_3}, \ \{x_2, x_3\} = \tfrac{\partial \varphi}{\partial x_1}, \ \{x_1, x_3\} = \tfrac{\partial \varphi}{\partial x_2}.$$

Note that the potential φ not only defines the Poisson relations but is also central $\{\varphi, \cdot\}$. Therefore the Poisson structure descends to the ring of functions on the surface $Z(\varphi) = \{(x_1, x_2, x_3) \in \mathbb{C}^3 | \varphi(x_1, x_2, x_3) = 0\}$. In other words, $A_\varphi = (\mathbb{C}[x_1, x_2, x_3], \{\cdot, \cdot\})/\langle \varphi \rangle$ is a Poisson algebra, so that the affine del Pezzo surface given by the zero set of φ, $Z(\varphi)$, is Spec(A_φ).

For our chosen φ we have

$$\{x_1, x_2\} = x_1 x_2 + \tfrac{\partial \phi_3}{\partial x_3}, \text{ and cyclic.}$$

To quantize, we define a suitable Lie algebra isomorphism:

$(A_\varphi, \{\cdot, \cdot\})$	$\longrightarrow$	$(\mathcal{A}_\Phi^\hbar, [\cdot, \cdot])$
commutative Poisson		non commutative
symmetric algebra		flat deformation

Example: For

$$\varphi = x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4,$$

we obtain the Askey-Wilson algebra [4]

$$\begin{gathered} q^{-\frac{1}{2}} x_1 x_2 - q^{-\frac{1}{2}} x_2 x_1 = (q^{-1} - q) x_3 - \Omega_3 x_3, \\ q^{-\frac{1}{2}} x_2 x_3 - q^{-\frac{1}{2}} x_3 x_2 = (q^{-1} - q) x_1 - \Omega_1 x_1, \\ q^{-\frac{1}{2}} x_3 x_1 - q^{-\frac{1}{2}} x_1 x_3 = (q^{-1} - q) x_2 - \Omega_2 x_2, \\ [\Omega_i, \cdot] = 0, \ i = 1, \dots, 4. \end{gathered}$$

Semiclassical limit: $\frac{[x_i, x_j]}{1-q} \underset{q \to 1}{\longrightarrow} \{x_i, x_j\}$ gives the character variety of a sphere with four boundaries.

The Painlevé–Sklyanin algebra

Definition. For any choice of the scalars

$$\alpha_i, \ \beta_i, \ a_i, \ b_i \in \mathbb{C}, \ i = 1, 2, 3,$$

such that α_i are not roots of unity, the *generalised Sklyanin-Painlevé algebra* is the non-commutative algebra with generators X_1, X_2, X_3 defined by the relations:

$$\begin{aligned} X_2X_3 - \alpha_1 X_3 X_2 - \beta_1 X_1^2 + a_1 X_1 + b_1 &= 0, \\ X_3X_1 - \alpha_2 X_1 X_3 - \beta_2 X_2^2 + a_2 X_2 + b_2 &= 0, \\ X_1X_2 - \alpha_3 X_2 X_1 - \beta_3 X_3^2 + a_3 X_3 + b_3 &= 0. \end{aligned}$$

We fully characterise for which cases the generalised Sklyanin-Painlevé algebra is a Calabi Yau algebra has nice properties [5]:

Theorem. For specific choices of the parameters as follows:

1. $\alpha_1 = \alpha_2 = \alpha_3 \neq 0$ and $(\alpha_1^3, \beta_1\beta_2\beta_3) \neq (-1, 1)$,
2. $(\alpha_1, \alpha_2, \alpha_3) \neq (0, 0, 0)$ and either $\beta_1 = \beta_2 = \alpha_1 - \alpha_2 = 0$ or $\beta_3 = \beta_1 = \alpha_3 - \alpha_1 = 0$ or $\beta_2 = \beta_3 = \alpha_2 - \alpha_3 = 0$,
3. $\beta_1 = \beta_2 = \beta_3 = 0$ and $(\alpha_1, \alpha_2, \alpha_3) \neq (0, 0, 0)$,

the generalised Sklyanin-Painlevé algebra is Calabi-Yau, has a Hilbert series with polynomial growth and is Koszul.

Note that for $a_i = b_i = 0$, $i = 1, 2, 3$, the Sklyanin–Painlevé algebra restricts to the so called Artin–Schelter–Tate–Sklyanin algebra with three generators. For $\beta_1 = \beta_2 = \beta_3 = 0$ and $(\alpha_1, \alpha_2, \alpha_3) \neq (0, 0, 0)$ this algebra produces the monodromy manifolds of the Painlevé differential equation in the semi-classical limit.

References

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Reviews

Books

Analysis in Euclidean Spaces (World Scientific 2023)
by Joaquim Bruna.
Reviewed by Sebastià Xambó.

This treatise, published in the series *Essential Textbooks in Mathematics*, is based on the notes written by the author in his long teaching experience. It is organized in 21 chapters, each with an introduction summarizing its contents, and an additional chapter with 122 miscellaneous exercises. The wealth of materials it presents, with emphasis on concepts and rigorous proofs, with many examples and exercises inserted throughout, are tied together by very careful reasoning all along, and are handy for a variety of purposes. It can be used as a textbook for mathematics and physics undergraduate students on subjects such as differentiation theory in several real variables, measure and integration in several real variables, ordinary differential equations, linear partial differential equations, vector analysis, and curves and surfaces. Graduate students may use this book for an introduction to geometric measure theory and integral geometry, as well as advanced topics in vector analysis. It is also very suitable for self-study, including revision of topics by readers wishing to refresh them.

One valuable feature of this text is that there are a good number of original presentations of classic results. Among them, the treatments of the Riemann integral (Chapter 11) and the Lebesgue integral (Chapter 12); the multidimensional version of the fundamental theorem of calculus (Section 12.8); and refined versions of the Helmholtz decomposition of vector fields (Chapter 20, particularly §20.4). No doubt this fresh view has also much interest in teaching. For example, the author's version of the fundamental theorem of calculus allows to give "the correct proof" of the change of variable theorem in an integral (Theorem 13.3), as well as the basic theorems of vector calculus, with minimal regularity assumptions (Chapters 16, 17, 18). Another example is Poincaré's lemma (Theorem 18.5). Actually, there are jewels that are rare to find elsewhere, as the phrasing in modern language of Darboux's proof of Liouville's theorem on the rigidity of conformal applications in dimension > 2 (§10.4), and the results of the French classics on systems of triply orthogonal surfaces and Lamé surfaces (Chapter 10).

Although the text is basically a book on differentiation and integration in several variables, there are many links and pointers to many other areas. For instance: analysis of one complex variable (§9.5 and §17.3); ordinary differential equations (§8.2); partial differential equations (§4.7); classical theory of curves and surfaces (Chapter 16); geometric analysis (Chapter 7); integral geometry (§15.2). In this way the author fosters a living image of the core subject matter within mathematical analysis that contrasts with isolated treatments found in the literature.

Above all, this volume provides a road to learn how to think mathematically in real analysis, and its applications in various fronts, with no more prerequisites than a basic course in linear algebra and a standard first-year calculus course in differentiation and integration.